WAKE ELECTROMAGNETIC FIELDS AND FORCES NEAR ($a \le r \le b$) A RESISTIVE OBJECT

$\Longrightarrow$ Part 1: Circular geometry and classical thick-wall regime

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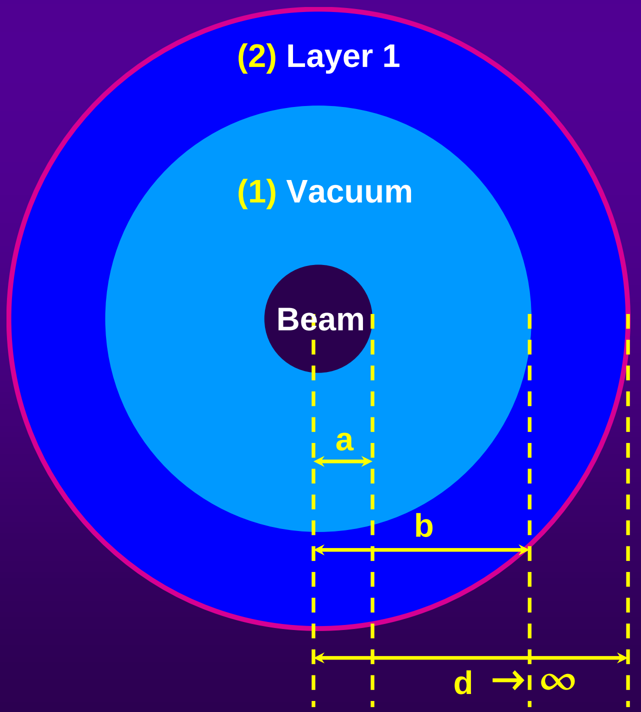
- **1st part of the answer to the question (from R. Assmann) about the EM fields near the collimator ($\Longrightarrow$ For SPS MD in 2004)**
- **Some more explanations about resistive-wall**



OUTLINE

- **Reminder: Impedance of a 1 m long round LHC graphite collimator**

There are Yokoya's factors to go from round to flat (π^2 / 12 and π^2 / 24)

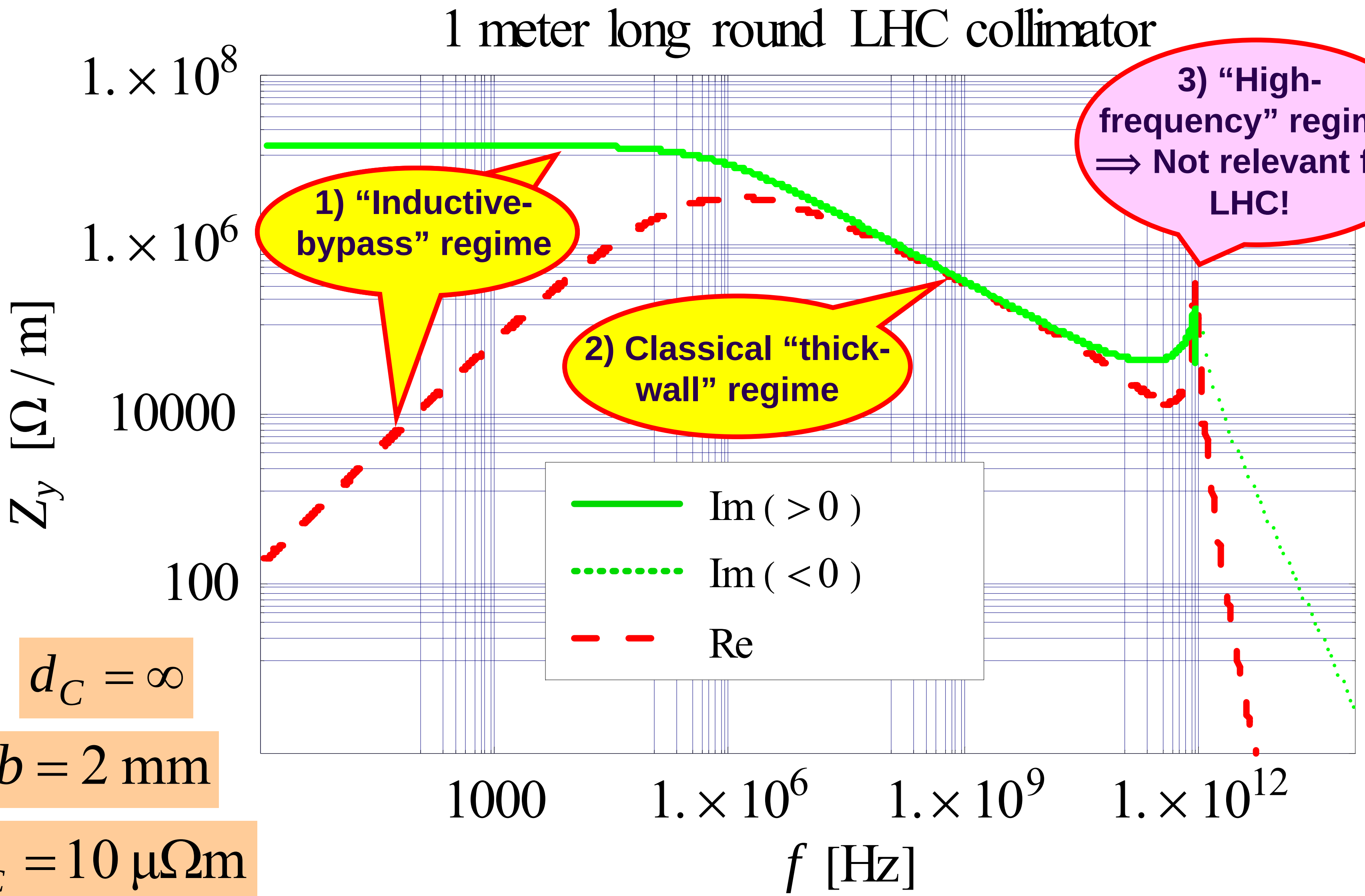
- **Impedance of a (1 layer) ceramic collimator**

- **Wake EM fields (6 components) and forces in the vacuum (between the beam and the chamber) deduced from Zotter2005's formalism**

  - **Perfect conductor $\Longrightarrow$ Same as Chao's book**
  - **Resistive object $\Longrightarrow$ Same as Chao except ("small" difference $\Longrightarrow$ Still to be checked) for E_r^{RW1} and E_ϑ^{RW1}**

- **Appendices: Link between wake fields, forces and impedances**

ZOTTER2005'S THEORY FOR 1 GRAPHITE COLLIMATOR



$d_C = \infty$

$b = 2$ mm

$\rho_C = 10\ \mu\Omega\mathrm{m}$

Interesting frequency range for LHC
$\Longrightarrow$ From few kHz to few GHz

SIMPLEST FORMULA FOR THE LHC COLLIMATOR TRANSVERSE IMPEDANCE (round case) (1/2)

$$Z_t^{RW1}(\omega) = \frac{j\,L\,Z_0}{\pi\,b^2} \times \frac{1}{1 - \frac{x_2}{\mu_r}\,\frac{K_1'(x_2)}{K_1(x_2)}}$$

with

$$\delta = \sqrt{\frac{2}{\mu_0\,\sigma\,\omega}}$$

$$x_2 = (1 + j)\,\frac{b}{\delta}$$

Modified Bessel function

$$\frac{K_1'(x_2)}{K_1(x_2)} = \begin{cases} -\dfrac{1}{x_2} & \text{if } |x_2| << 1 \\ -1 & \text{if } |x_2| >> 1 \end{cases}$$

$\Rightarrow$

$$Z_t^{RW1}(\omega) \xrightarrow[\omega \to 0]{} \frac{j\,L\,Z_0}{\pi\,b^2} \times \frac{1}{1 + \frac{1}{\mu_r}}$$

$\Rightarrow$

$$Z_t^{RW1}(\omega) = (1 + j)\,\frac{L\,Z_0\,\mu_r\,\delta}{2\,\pi\,b^3}$$

Classical "thick-wall" regime

$$\mu_r = \frac{\mu}{\mu_0}$$

SIMPLEST FORMULA FOR THE LHC COLLIMATOR TRANSVERSE IMPEDANCE (round case) (2/2)

The maximum of the real part is reached when $\mathrm{Re}\left[x_2\right] \approx 1$

It is a broad maximum

$$\Leftrightarrow \quad \delta \approx b$$

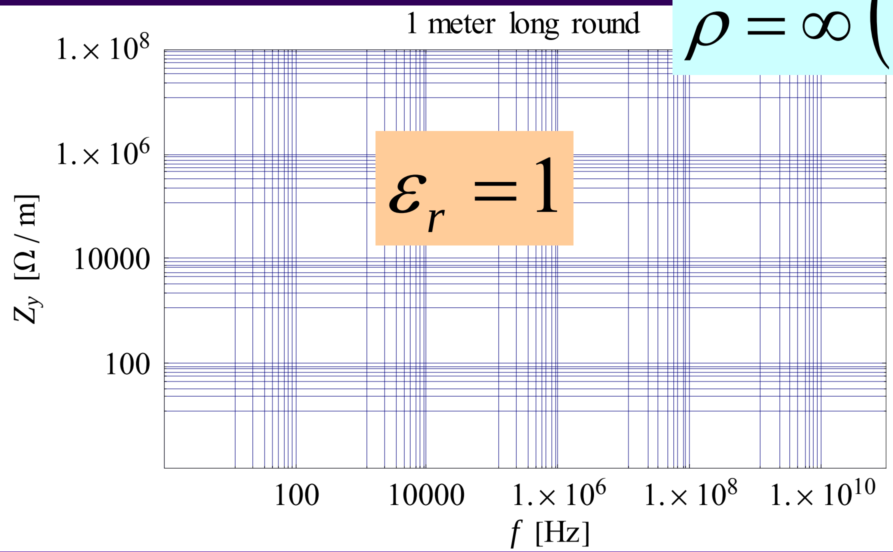
$$\Longrightarrow \quad f_{\max \mathrm{Re}} \approx \frac{\rho}{b^2} \times \frac{1}{\pi \, \mu_0}$$

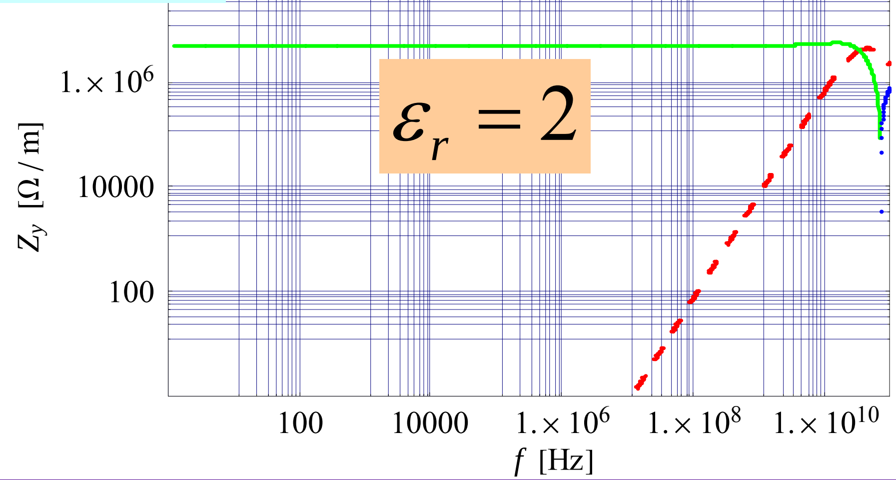
This scaling was also found analytically using the approximated model of L. Vos (as said in the LHC Design Report, p. 100)

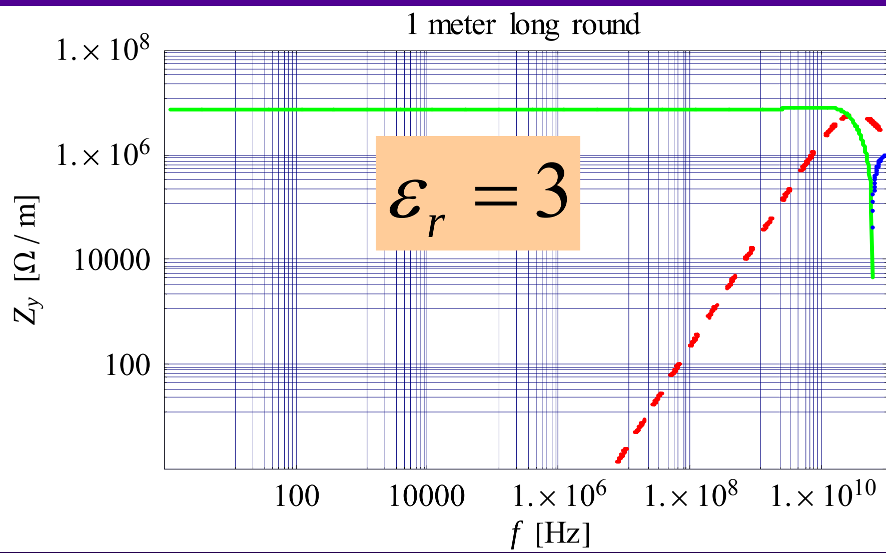
N.A.: $f_{\max \mathrm{Re}} \approx 0.6 \ \mathrm{MHz}$

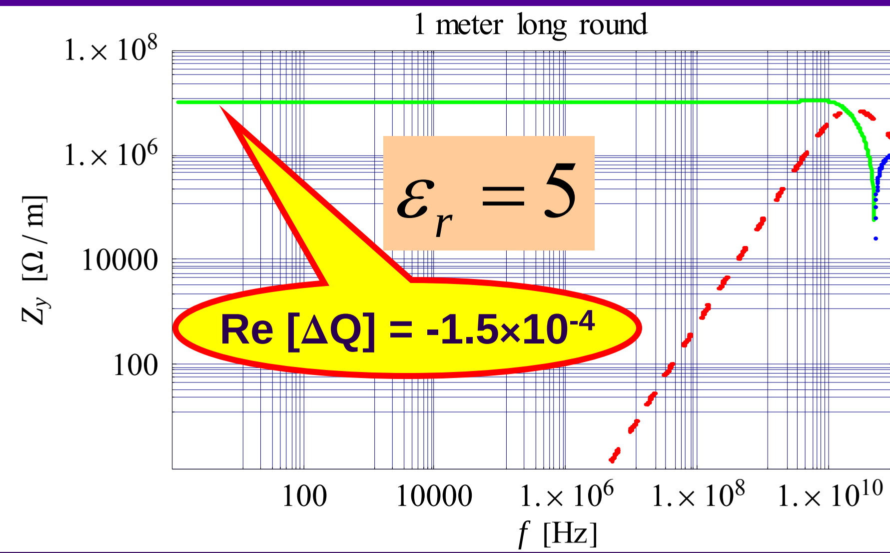
TRANSVERSE IMPEDANCE OF A CERAMIC (1 layer) COLLIMATOR (1/2)

$$\rho = \infty \left(10^{20}\ \Omega\text{m} \right)$$



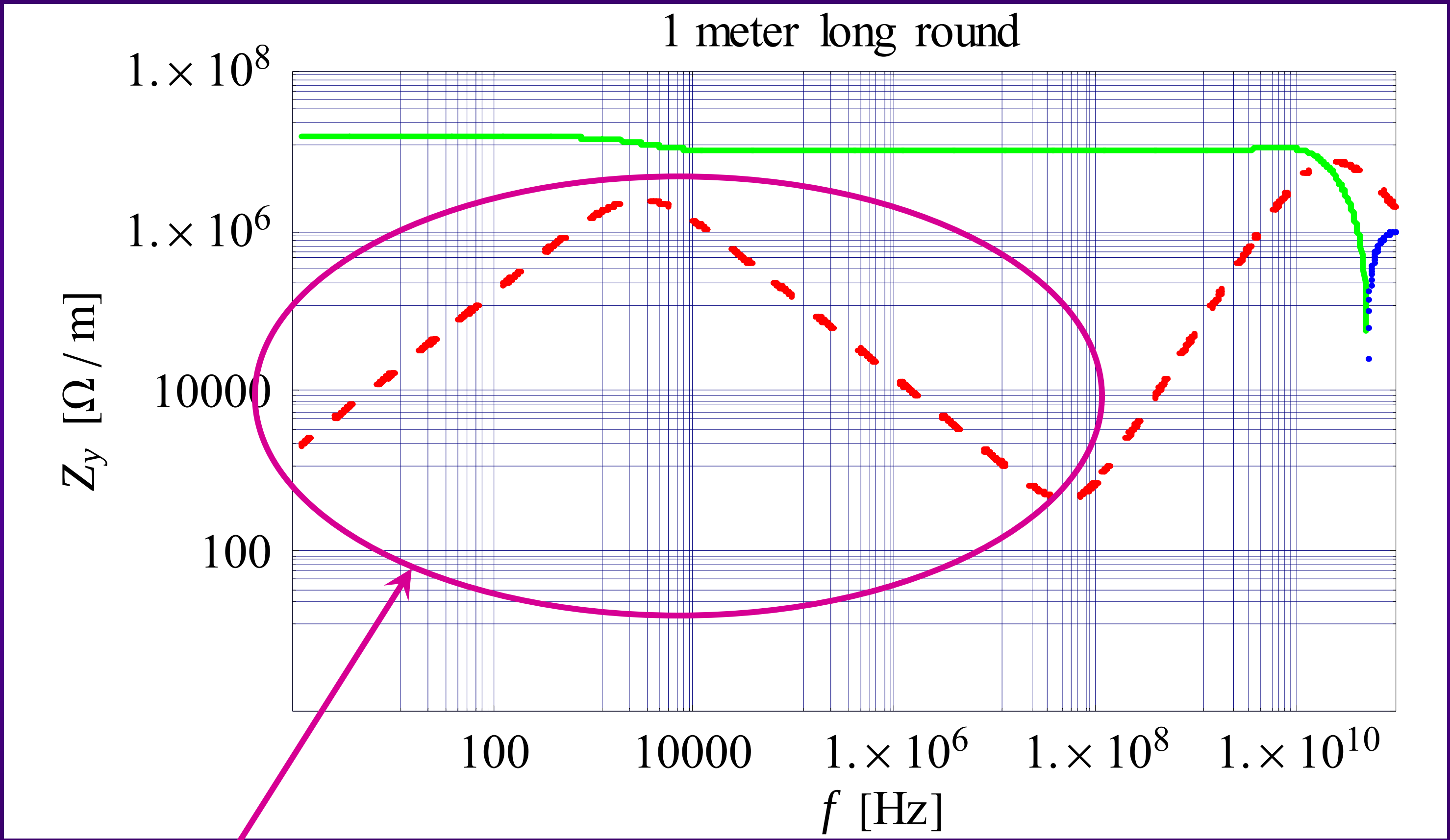






TRANSVERSE IMPEDANCE OF A CERAMIC (1 layer) COLLIMATOR (2/2)

$\varepsilon_r = 5$ $\rho = 10^6\ \Omega\text{m}$



Preliminary! $\Longrightarrow$ To be checked!

SOURCE CHARGE DENSITY USED FOR THE COMPUTATIONS

- A macro-particle of charge $Q = N_b \, e$ is assumed to move along the pipe (in the *s* - direction) with an offset $r = a$ in the $\vartheta = 0$ direction and with velocity $\upsilon = \beta \, c$

$\Longrightarrow$ The charge density can be written

$$\rho\left(r, \vartheta, s; t\right) = \sum_{m=0}^{\infty} \frac{P_m \cos\left(m \vartheta\right)}{\pi \, a^{m+1} \left(1 + \delta_{m0}\right)} \, \delta\left(r - a\right) \delta\left(s - \upsilon \, t\right)$$

where $P_m = Q \, a^m$ is the *m*th multipole moment

- The cylindrical coordinate system $\left(\vec{r}, \vec{\vartheta}, \vec{s}\right)$ is used

- Numerical values are given for the 2004 SPS experiment $N_b = 1.15 \times 10^{11}$ p/b $r = b = 2$ mm $\vartheta = 0$

REMINDER: Perfectly Conducting wall (1/3)

***m* = 0**

Used to compute the longitudinal impedance

$$E_\vartheta^{PC0} = B_r^{PC0} = B_s^{PC0} = 0$$

$$E_s^{PC0} = \frac{Q}{2\pi\varepsilon_0\gamma^2}\ln\left(\frac{b}{r}\right)\delta'(s-\upsilon t) \xrightarrow[\gamma\to\infty]{} 0$$

$$E_s^{PC0} = 0$$

$$E_r^{PC0} = \frac{Q}{2\pi\varepsilon_0 r}\,\delta(s-\upsilon t)$$

$$E_r^{PC0} \approx 1.7\times10^5\;\delta(s-\upsilon t)$$

$$B_\vartheta^{PC0} = \frac{\beta}{c}\,E_r^{PC0}$$

$$B_\vartheta^{PC0} \approx 6\times10^{-4}\;\delta(s-\upsilon t)$$

***m* = 1**

Used to compute the transverse impedance

$$B_s^{PC1} = 0$$

$$E_s^{PC1} = 0$$

$$E_s^{PC1} = \frac{P_1\cos(\vartheta)}{2\pi\varepsilon_0\gamma^2}\left[\frac{1}{r}-\frac{r}{b^2}\right]\delta'(s-\upsilon t) \xrightarrow[\gamma\to\infty]{} 0$$

$$E_r^{PC1} = \frac{P_1\cos(\vartheta)}{2\pi\varepsilon_0}\left[\frac{1}{r^2}+\frac{1}{b^2}\right]\delta(s-\upsilon t)$$

$$E_r^{PC1} \approx 0.9\times10^5\;\delta(s-\upsilon t)$$

$$E_\vartheta^{PC1} = \frac{P_1\sin(\vartheta)}{2\pi\varepsilon_0}\left[\frac{1}{r^2}-\frac{1}{b^2}\right]\delta(s-\upsilon t)$$

$$E_\vartheta^{PC1} = 0$$

$$B_\vartheta^{PC1} = \frac{\beta}{c}\,E_r^{PC1} \qquad B_r^{PC1} = -\frac{\beta}{c}\,E_\vartheta^{PC1}$$

$$B_\vartheta^{PC1} \approx 3\times10^{-4}\;\delta(s-\upsilon t)$$

$$B_r^{PC1} = 0$$

REMINDER: Perfectly Conducting wall (2/3)

Force on a particle with charge *q*

$$\vec{F} = q\left[E_s\,\vec{s} + \left(E_r - \upsilon\,B_\vartheta\right)\vec{r} + \left(E_\vartheta + \upsilon\,B_r\right)\vec{\vartheta}\right]$$

***m* = 0**

$$F_s^{PC0} = \frac{q\,Q}{2\pi\,\varepsilon_0\,\gamma^2}\ln\left(\frac{b}{r}\right)\delta'\left(s - \upsilon t\right) \xrightarrow[\gamma\to\infty]{} 0$$

$$F_r^{PC0} = \frac{q\,Q}{2\pi\,\varepsilon_0\,r\,\gamma^2}\,\delta\left(s - \upsilon t\right)$$

$$F_\vartheta^{PC0} = 0$$

***m* = 1**

$$F_s^{PC1} = \frac{q\,P_1\cos\left(\vartheta\right)}{2\pi\,\varepsilon_0\,\gamma^2}\left[\frac{1}{r} - \frac{r}{b^2}\right]\delta'\left(s - \upsilon t\right) \xrightarrow[\gamma\to\infty]{} 0$$

$$F_r^{PC1} = \frac{q\,P_1\cos\left(\vartheta\right)}{2\pi\,\varepsilon_0\,\gamma^2}\left[\frac{1}{r^2} + \frac{1}{b^2}\right]\delta\left(s - \upsilon t\right)$$

$$F_\vartheta^{PC1} = \frac{q\,P_1\sin\left(\vartheta\right)}{2\pi\,\varepsilon_0\,\gamma^2}\left[\frac{1}{r^2} - \frac{1}{b^2}\right]\delta\left(s - \upsilon t\right)$$

REMINDER: Perfectly Conducting wall (3/3)

m = 0

$$Z_l^{PC0}(\omega) = -j\,\frac{L\,\omega\,Z_0}{2\pi\,c\,\beta^2\gamma^2}\ln\left(\frac{b}{a}\right)$$

$$W_l^{PC0}(\tau) = -\frac{L\,Z_0}{2\pi\,c\,\beta^2\gamma^2}\ln\left(\frac{b}{a}\right)\delta'(\tau)$$

Behind the bunch

For L = 2 π R

$$Z_l^{PC0}(\omega) = -j\,\frac{\omega\,Z_0}{\omega_0\,\beta\,\gamma^2}\ln\left(\frac{b}{a}\right)$$

$$W_l^{PC0}(\tau) = -\frac{Z_0}{\omega_0\,\beta\,\gamma^2}\ln\left(\frac{b}{a}\right)\delta'(\tau)$$

m = 1

$$Z_t^{PC1}(\omega) = -j\,\frac{L\,Z_0}{2\pi\,\beta\,\gamma^2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)$$

$$W_t^{PC1}(\tau) = -\frac{L\,Z_0}{2\pi\,\beta\,\gamma^2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\delta(\tau)$$

For L = 2 π R

$$Z_t^{PC1}(\omega) = -j\,\frac{R\,Z_0}{\beta\,\gamma^2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)$$

$$W_t^{PC1}(\tau) = -\frac{R\,Z_0}{\beta\,\gamma^2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\delta(\tau)$$

CASE OF A RESISTIVE OBJECT (1/4)

$\gamma \to \infty$

$z = s - c\,t$

⟹ z > 0 is ahead of the beam and z < 0 is behind the beam

m = 0

$$E_\vartheta^{RW0} = B_r^{RW0} = B_s^{RW0} = 0$$

$$E_s^{RW0} = \frac{Q\,c\,\sqrt{Z_0}}{4\,\pi^{3/2}\,b\,\sqrt{\sigma}\,|\,z\,|^{3/2}}$$

$$E_s^{RW0} \approx \frac{8}{|\,z\,|^{3/2}}$$

$$E_r^{RW0} = -\frac{3\,Q\,r\,c\,\sqrt{Z_0}}{16\,\pi^{3/2}\,b\,\sqrt{\sigma}\,|\,z\,|^{5/2}}$$

$$E_r^{RW0} \approx -\frac{0.01}{|\,z\,|^{5/2}}$$

$$B_\vartheta^{RW0} = \frac{E_r^{RW0}}{c}$$

$$c\,B_\vartheta^{RW0} \approx -\frac{0.01}{|\,z\,|^{5/2}}$$

m = 1

$$E_s^{RW1} \approx \frac{4}{|\,z\,|^{3/2}}$$

$$E_s^{RW1} = \frac{P_1 \cos(\vartheta)\,c\,r\,\mu_r\,\sqrt{Z_0}}{2\,\pi^{3/2}\,b^3\,\sqrt{\sigma}\,|\,z\,|^{3/2}}$$

E_r^{RW1} E_ϑ^{RW1}

Different from Chao page 54 (see next slide)!

$$B_s^{RW1} = -\frac{P_1 \sin(\vartheta)\,r\,\mu_r\,\sqrt{Z_0}}{2\,\pi^{3/2}\,b^3\,\sqrt{\sigma}\,|\,z\,|^{3/2}}$$

$$B_s^{RW1} = 0$$

$$E_\vartheta^{RW1} + c\,B_r^{RW1} = -\frac{P_1 \sin(\vartheta)\,c\,\mu_r\,\sqrt{Z_0}}{\pi^{3/2}\,b^3\,\sqrt{\sigma}\,|\,z\,|^{1/2}}$$

$$E_\vartheta^{RW1} + c\,B_r^{RW1} = 0$$

$$E_r^{RW1} - c\,B_\vartheta^{RW1} = \frac{P_1 \cos(\vartheta)\,c\,\mu_r\,\sqrt{Z_0}}{\pi^{3/2}\,b^3\,\sqrt{\sigma}\,|\,z\,|^{1/2}}$$

$$E_r^{RW1} - c\,B_\vartheta^{RW1} \approx \frac{4\times10^3}{|\,z\,|^{1/2}}$$

CASE OF A RESISTIVE OBJECT (2/4)

I have (at the moment…) for $m = 1$

Chao finds the same result for $r = b$ and $\theta = 0$

$\mu_r = 1$ for Chao

$$E_\vartheta^{RW1} = 0$$

$$E_\vartheta^{RW1} = - \frac{3 \, P_1 \sin(\vartheta) \, c \, \mu_r \sqrt{Z_0}}{16 \, \pi^{3/2} \, b^3 \sqrt{\sigma} \, |z|^{5/2}} \, r^2$$

Chao has ($r^2 - b^2$) instead of r^2

Chao finds a result 2 times bigger

$$E_r^{RW1} \approx \frac{3 \times 10^{-3}}{|z|^{5/2}}$$

$$E_r^{RW1} = - \frac{3 \, P_1 \cos(\vartheta) \, c \, \mu_r \sqrt{Z_0}}{16 \, \pi^{3/2} \, b^3 \sqrt{\sigma} \, |z|^{5/2}} \, r^2$$

Chao has ($r^2 + b^2$) instead of r^2

CASE OF A RESISTIVE OBJECT (3/4)

***m* = 0**

$$F_s^{RW0} = \frac{q\, Q\, c\, \sqrt{Z_0}}{4\, \pi^{3/2}\, b\, \sqrt{\sigma}\, |\, z\, |^{3/2}}$$

$$F_r^{RW0} = F_\vartheta^{RW0} = 0$$

***m* = 1**

$$F_s^{RW1} = \frac{q\, P_1 \cos(\vartheta)\, c\, r\, \mu_r\, \sqrt{Z_0}}{2\, \pi^{3/2}\, b^3\, \sqrt{\sigma}\, |\, z\, |^{3/2}}$$

$$F_r^{RW1} = \frac{q\, P_1 \cos(\vartheta)\, c\, \mu_r\, \sqrt{Z_0}}{\pi^{3/2}\, b^3\, \sqrt{\sigma}\, |\, z\, |^{1/2}}$$

$$F_\vartheta^{RW1} = -\frac{q\, P_1 \sin(\vartheta)\, c\, \mu_r\, \sqrt{Z_0}}{\pi^{3/2}\, b^3\, \sqrt{\sigma}\, |\, z\, |^{1/2}}$$

CASE OF A RESISTIVE OBJECT (4/4)

m = 0

$$Z_l^{RW0}(\omega) = (1 + j)\frac{L}{2\pi b}\sqrt{\frac{\omega Z_0}{2 c \sigma}}$$

$$W_l^{RW0}(\tau) = -\frac{L}{4\pi^{3/2} b}\sqrt{\frac{Z_0}{c \sigma}} \times \frac{1}{\tau^{3/2}}$$

m = 1

$$Z_t^{RW1}(\omega) = (1 + j)\frac{L Z_0}{\pi b^3}\frac{\mu_r}{\sqrt{2 \mu_0 \sigma \omega}}$$

$$W_t^{RW1}(\tau) = \frac{L \mu_r}{\pi^{3/2} b^3}\sqrt{\frac{c Z_0}{\sigma}} \times \frac{1}{\tau^{1/2}}$$

APPENDIX A: WAKE FORCES AND WAKE FIELDS

- ***m* = 0**

$$\int_{-L/2}^{L/2} F_r^0 \, ds = \int_{-L/2}^{L/2} F_\vartheta^0 \, ds = 0$$

$$\int_{-L/2}^{L/2} F_s^0 \, ds = -q \, P_0 \, W_0'(z)$$

Longitudinal wake function

Transverse wake function

- ***m* = 1**

$$\int_{-L/2}^{L/2} F_r^1 \, ds = q \, P_1 \cos(\vartheta) W_1(z)$$

$$\int_{-L/2}^{L/2} F_\vartheta^1 \, ds = -q \, P_1 \sin(\vartheta) W_1(z)$$

$$\int_{-L/2}^{L/2} F_s^1 \, ds = -q \, P_1 \cos(\vartheta) r \, W_1'(z)$$

APPENDIX B: WAKE FIELDS AND IMPEDANCES (1/2)

- **Fourier transforms**

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, dt \qquad f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\omega)\, e^{j\omega t}\, d\omega$$

- **Longitudinal plane**

$$Z_l^0(\omega) = \int_{-\infty}^{+\infty} W_l^0(t)\, e^{-j\omega t}\, dt \qquad W_l^0(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} Z_l^0(\omega)\, e^{j\omega t}\, d\omega$$

- **Transverse plane**

$$Z_t^1(\omega) = j \int_{-\infty}^{+\infty} W_t^1(t)\, e^{-j\omega t}\, dt \qquad W_t^1(t) = \frac{-j}{2\pi} \int_{-\infty}^{+\infty} Z_t^1(\omega)\, e^{j\omega t}\, d\omega$$

APPENDIX B: WAKE FIELDS AND IMPEDANCES (2/2)

- **As the conductivity, permittivity and permeability of a material depend in general on frequency, it is usually better (or easier) to treat the problem in the frequency domain**

  $\Longrightarrow$ The impedance is computed first

  $\Longrightarrow$ The wake-field is deduced by inverse Fourier transform (analytically in some cases but most of the time it has to be done numerically)